Partitioned Matrices
Reading: Lay 2.4

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Say we have a matrix A. One way we have often thought about A is as a collection of vectors, the columns of A. In this lecture, we talk about other ways to break a matrix up into pieces – "blocks" or "partitions".

1 Partitions

We will consider matrices broken up into **partitions** or **blocks** by horizontal and vertical dividing lines–that is, lines drawn vertically and horizontally across the matrix to "break it up". For instance, the matrix

$$A = \begin{bmatrix} 1 & 3 & -6 & 2 \\ 2 & 1 & 0 & -1 \\ 3 & 3 & 4 & 4 \end{bmatrix}$$

can be written as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where

$$A_{11} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix},\ A_{12} = \begin{bmatrix} -6 & 2 \\ 0 & -1 \end{bmatrix},\ A_{21} = \begin{bmatrix} 3 & 3 \end{bmatrix},\ A_{22} = \begin{bmatrix} 4 & 4 \end{bmatrix}.$$

2 Operations on Partitioned Matrices

2.1 Addition, Scalars

If two matrices A and B are the same size and partitioned in exactly the same way, then it is easy to see that we can partition the sum $A+B$ in the same way, and that each block of $A+B$ is equal to the corresponding block of A plus the corresponding block of B. For instance, if A and B are the same size and partitioned the same way with

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

then

$$A + B = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{bmatrix}.$$

Similarly, if c is a scalar, then each block of cA is equal to c times the corresponding block of A.

2.2 Matrix Multiplication with Partitions

Given two partitioned matrices A and B such that the product AB is defined, we want to express the product AB in terms of the blocks of A and B. It turns out that we can actually multiply the partitioned matrices as though the blocks were scalars, assuming that the blocks are the correct sizes.

The important thing for defining the product AB of partitioned matrices is that that the columns of A are broken up in the same way as the rows of B. We call such a pair of partitions **conformable**. For instance, if we have the matrices

$$A = \begin{bmatrix} 1 & 3 & -6 & 2 \\ 2 & 1 & 0 & -1 \\ 3 & 3 & 4 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \end{bmatrix}$$

partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where

$$A_{11} = \begin{bmatrix} 1 & 3 & -6 \\ 2 & 1 & 0 \end{bmatrix},\ A_{12} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},\ A_{21} = \begin{bmatrix} 3 & 3 & 4 \end{bmatrix},\ A_{22} = \begin{bmatrix} 4 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix},\ B_2 = \begin{bmatrix} 4 & 4 & 4 \end{bmatrix},$$

then the partitions of A and B are conformable? Why? Because the columns of A are broken up into 3 columns followed by 1 column, and the rows of B are broken up into 3 rows followed by 1 row.

Now, let's compute. Note that

$$AB = \begin{bmatrix} -3 & -3 & -3 \\ 0 & 0 & 0 \\ 37 & 37 & 37 \end{bmatrix}.$$

Imagine that we tried to multiply A and B treating the blocks as just numbers. We would get

$$AB = \begin{bmatrix} A_1 1 B_1 + A_{12} B_2 \\ A_{21} B_1 + A_{22} B_2 \end{bmatrix}.$$

On the other hand, we get for the products of blocks

$$A_{11} B_1 = \begin{bmatrix} -11 & -11 & -11 \\ 4 & 4 & 4 \end{bmatrix},\ A_{12} B_2 = \begin{bmatrix} 8 & 8 & 8 \\ -4 & -4 & -4 \end{bmatrix},$$

$$A_{21} B_1 = \begin{bmatrix} 21 & 21 & 21 \end{bmatrix},\ A_{22} B_2 = \begin{bmatrix} 16 & 16 & 16 \end{bmatrix}.$$

So we have

$$\begin{bmatrix} A_1 1 B_1 + A_{12} B_2 \\ A_{21} B_1 + A_{22} B_2 \end{bmatrix} = \begin{bmatrix} -3 & -3 & -3 \\ 0 & 0 & 0 \\ 37 & 37 & 37 \end{bmatrix}.$$

This agrees with what we computed using the traditional method for calculating AB!

This method works for any matrices with conformable partitions. Using this method lets us think about matrix multiplication in another way, as shown in the following theorem.

Theorem 2.1. *If A is $m \times n$ and B is $n \times p$, let us denote the kth column of A by* $\mathrm{col}_k(A)$ *and the kth row of B by* $\mathrm{row}_k(B)$*. Then we have*

$$\begin{aligned} AB &= \begin{bmatrix} \mathrm{col}_1(A) & \mathrm{col}_2(A) & \ldots & \mathrm{col}_n(A) \end{bmatrix} \begin{bmatrix} \mathrm{row}_1(B) \\ \mathrm{row}_2(B) \\ \vdots \\ \mathrm{row}_n(B) \end{bmatrix} \\ &= \mathrm{col}_1(A)\mathrm{row}_1(B) + \ldots + \mathrm{col}_n(A)\mathrm{row}_n(B). \end{aligned}$$

3 Inverting Partitioned Matrices

Recall that a diagonal matrix is a matrix whose entries are zero except on the main diagonal (the diagonal running from the upper left of the matrix to the lower right). We call a matrix **block diagonal** if it is partitioned and all of the block matrices which are not on the main (block) diagonal are zero.

In certain special cases it is possible to find nice forms for the inverse of a block diagonal matrix in terms of its blocks. I will work in class Lay's example 5, and also his practice problem 1. You should look at these to get a feeling for how this works.